SUBGRAPH POSETS, PARTITION LATTICES, GRAPH POLYNOMIALS AND RECONSTRUCTION

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Abstract. Two objects of fundamental importance in the study of graph polynomials are the poset of induced subgraphs of a graph and the lattice of its connected partitions. In my earlier paper I showed that several invariants of a graph can be computed from the isomorphism class of its poset of non-empty induced subgraphs, (that is, subgraphs themselves are not required). In this paper I will prove that the (abstract and folded) connected partition lattice of a graph can be constructed from its abstract poset of induced subgraphs. I will also prove that, except when the graph is a star or a disjoint union of edges, the abstract induced subgraph poset of the graph can be constructed from its abstract folded connected partition lattice. The first construction implies that if a graph polynomial has an expansion on the connected partition lattice then it is reconstructible from the isomorphism class of the poset of non-empty induced subgraphs. Examples of such polynomials are the chromatic symmetric function and the symmetric Tutte polynomial. The second construction implies that a tree can be reconstructed from the isomorphism class of its folded connected partition lattice. This is a possible line of attack for Stanley's question whether the chromatic symmetric function is a complete invariant of trees. I then show that the symmetric Tutte polynomial of a tree can be computed from the chromatic symmetric function of the tree, thus showing that a question of Noble and Welsh is equivalent to Stanley's question about the chromatic symmetric function of trees. The paper also develops edge reconstruction theory on the edge subgraph poset, and its relation with Lovász's homomorphism cancellation laws. In particular I present a conjecture generalising Lovász's homomorphism cancellation laws, and show that it is weaker than the edge reconstruction conjecture. A characterisation of a family of graphs that cannot be constructed from their abstract edge subgraph posets is also presented.

Date: 10 September, 2006.

2000 *Mathematics Subject Classification.* Primary: 05C60, 05E05.

Key words and phrases. reconstruction, chromatic symmetric function, partition lattices.

Contents

1. Introduction

Only finite simple graphs are considered in this paper. Let G be a graph. Let $\mathcal{P}_v(G) = \{G_1, G_2, \ldots, G_n\}$ be its set of mutually non-isomorphic non-empty unlabelled induced subgraphs. Let $\mathcal{P}_v(G)$ be partially ordered by defining $G_i \preceq_v G_j$ if G_i is isomorphic to an induced subgraph of G_j for any two G_i and G_j. This poset is edge labelled by associating with each pair of graphs $G_i \preceq_v G_j$ the number of induced subgraphs of G_j that are isomorphic to G_i. Such an edge labelled poset is called the *induced subgraph poset* of G. In [13] I showed that vertex reconstruction conjecture is true if and only if all graphs can be constructed from their (abstract) induced subgraph posets. I demonstrated that several invariants of a graph can be calculated from its (abstract) induced subgraph poset. Here by *abstract* induced subgraph poset we mean the *isomorphism class* of the induced subgraph poset. In other words, to compute the graph invariants, we do not require the induced subgraphs, but require only the isomorphism class of the induced subgraph poset. Thus many classic results of Tutte [14](e.g., reconstruction of the number of spanning trees, the number of hamiltonian cycles, the chromatic polynomial, the characteristic polynomial, the rank polynomial, the number of unicyclic subgraphs containing a

cycle of a specified length, etc.) were proved to be reconstructible from the abstract induced subgraph poset. Computing the above invariants requires counting certain spanning subgraphs, and the proofs involve constructions that are closely related to the connected partition lattice of the graph. I commented in Section 5 of [13] that understanding a relationship between the induced subgraph poset of a graph and its lattice of connected partitions would be interesting since certain important graph invariants, for example, Stanley's chromatic symmetric function, have nice expansions on the connected partition lattice. I also indicated that Kocay's lemma in graph reconstruction theory could be a tool in understanding such a relationship. One purpose of this paper is to clarify this relationship. I will define the notion of an edge labelled folded connected partition lattice. That is, a lattice will be defined on the set of mutually non-isomorphic spanning subgraphs of a graph that are induced by connected partitions of the graph. This lattice will be appropriately edge labelled. I will present two constructions that allow us to construct the (abstract) induced subgraph poset of a graph and the (abstract) folded connected partition lattice of the graph from each other (except when the graph is a star or a disjoint union of edges). I will also prove that the symmetric Tutte polynomial introduced by Stanley, which specialises to the Tutte polynomial as well as the chromatic symmetric function, is vertex reconstructible in a stronger sense: the symmetric Tutte polynomial of a graph can be computed from the isomorphism class of its induced subgraph poset. Another motivation of this paper is a question of Stanley regarding the chromatic symmetric function: can non-isomorphic trees have the same chromatic symmetric function? In [13], I proved that a tree can be reconstructed from the isomorphism class of its induced subgraph poset. One of the constructions presented here implies that a tree can be constructed from its abstract folded connected partition lattice. In another construction, it will be demonstrated that the symmetric Tutte polynomial of a tree can be obtained from its chromatic symmetric function, thus showing that the questions raised by Stanley, and by Noble & Welsh are in fact equivalent.

In Section 5, I will develop edge reconstruction theory in an analogously defined framework of the edge subgraph poset of a graph. I will prove that the edge reconstruction conjecture is true if and only if all graphs, except the ones in a completely characterised class, are reconstructible from their (abstract) edge subgraph posets. Developing the edge reconstruction theory on the edge subgraph poset seems to be surprisingly more difficult than the vertex reconstruction theory

on the induced subgraph poset. A relationship between the edge reconstruction conjecture and a statement more general than Lovász's homomorphism cancellation laws [5] is derived in Section 6.

2. Preliminaries

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices, the number of edges and the number of components of G are denoted by $v(G)$, $e(G)$, and $c(G)$, respectively. A *null graph* Φ is a graph with empty vertex set, and an *empty graph* is a graph with empty edge set.

When F is a subgraph of G, we write $F \subseteq G$, and when F is a proper subgraph of G, we write $F \subsetneq G$. Isomorphism of two graphs G and H is denoted by $G \cong H$. A graph G with components isomorphic to H_i; $1 \leq i \leq p$, some of which possibly isomorphic, is written in a product notation as $G \cong \prod_{i=1}^{p} H_i$. If H_i are mutually non-isomorphic graphs, and G has k_i components isomorphic to H_i; $1 \leq i \leq p$, then G is written as $G \cong \prod_i H_i^{k_i}$.

The subgraph of G induced by $X \subseteq V(G)$ is the subgraph whose vertex set is X and whose edge set contains all the edges having both end vertices in X. It is called a vertex induced subgraph or simply an induced subgraph or a vertex-subgraph. It is denoted by $G[X]$. The subgraph of G induced by $V(G) - X$ is denoted by $G - X$, or simply $G - u$ if $X = \{u\}$. The number of vertex-subgraphs of G that are isomorphic to F is denoted by $\#(F \xrightarrow{v} G)$.

Let $\mathcal{X} = \{X_1, X_2, \ldots, X_p\}$ be a family of non-empty subsets of $V(G)$. The graph $\cup_{i=1}^{p} G[X_i]$ is denoted by $G[X_1, X_2, \ldots, X_p]$ or $G[\mathcal{X}]$. It is referred to as the subgraph of G induced by $\mathcal{X}$. When the sets X_i are mutually disjoint, the subgraph is called a π-subgraph of G. The number of π-subgraphs of G that are isomorphic to F is denoted by $\#(F \xrightarrow{\pi} G)$. We will be interested in π-subgraphs that span $V(G)$ and incidence relationships between them.

Let S be a subset of $E(G)$. The subgraph of G induced by S is the subgraph with edge set S and vertex set consisting of end vertices of edges in S. It is called an *edge induced subgraph* or simply an *edge-subgraph.* It is denoted by $G[S]$. The number of edge subgraphs of G that are isomorphic to F is denoted by $\#(F \xrightarrow{e} G)$.

The sets of integers, non-negative integers and positive integers are $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{P}$, respectively, and the set $\{1, 2, \ldots, n\}$ is denoted by $[n]$.

The two objects of interest in this paper are the poset of non-empty induced subgraphs of a graph and the connected partition lattice of a graph (and their variations to be described below).

2.1. **The induced subgraph poset of a graph.**

This was introduced above, but it is more formally defined here.

Definition 1. Given a graph G, let $\mathcal{P}_v(G) = \{G_i; 0 \leq i \leq n\}$ be a set of graphs such that

(1) the graphs $G_i; 0 \leq i \leq n$ are mutually non-isomorphic,
(2) $e(G_i) > 0$ for $i > 0$, and $G_0 \cong K_1$,
(3) every G_i is isomorphic to an induced subgraph of G, and every induced subgraph of G having a non-empty edge set is isomorphic to a unique G_i,
(4) $v(G_i) \leq v(G_{i+1})$ for $0 \leq i < n$.

Let $\preceq_v$ be a partial order relation on $\mathcal{P}_v(G)$ defined by $G_i \preceq_v G_j$ if G_i is isomorphic to an induced subgraph of G_j. The partially ordered set $(\mathcal{P}_v(G), \preceq_v)$ is *edge labelled* by associating with each related pair $G_i \preceq_v G_j$ of graphs the number $\#(G_i \xrightarrow{v} G_j)$. The edge labelled poset itself is also denoted by $\mathcal{P}_v(G)$, and will be simply referred to as the *induced subgraph poset* of G.

Note that although we call it the induced subgraph poset, it is not the poset of all induced subgraphs of G, and it is really more than a partially ordered set.

We say that two induced subgraph posets are *isomorphic* if they are isomorphic as posets, and there is an isomorphism between them that preserves the edge labels. Formally, $\mathcal{P}_v(G)$ is isomorphic to $\mathcal{P}_v(H)$ if there is a bijection π from $\mathcal{P}_v(G)$ to $\mathcal{P}_v(H)$ such that $\#(G_i \xrightarrow{v} G_j) = \#(\pi(G_i) \xrightarrow{v} \pi(G_j))$ for all G_i and G_j in $\mathcal{P}_v(G)$. The *unlabelled or abstract* induced subgraph poset of G is the isomorphism class of $\mathcal{P}_v(G)$. An isomorphism from an induced subgraph poset to itself is called an *automorphism* of the induced subgraph poset.

2.2. **The connected partition lattice of a graph.**

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of $n \in \mathbb{P}$. We write $\lambda \vdash n$, and denote its length by $l = l(\lambda)$. Although λ is written as a tuple, the order of λ_i is not important. When $\pi = \{X_i; i \in [l]\}$ is a partition of a non-empty set V, we write $\pi \vdash V$. An integer partition associated with π is the partition $\lambda(\pi) = (|X_1|, |X_2|, \ldots, |X_l|)$ of $|V|$. It is called the *type* of π.

Let $\pi = \{X_i; i \in [p]\}$ and $\sigma = \{Y_i; i \in [q]\}$ be two partitions of a set V. Define $\sigma \preceq_p \pi$ if each Y_i is a subset of some X_j. This relation defines a lattice on the set of partitions of V. It is called the partition lattice of V, and is denoted by $\prod(V)$.

Given a graph G, a partition π of $V(G)$ is called a *connected partition* if each cell of π induces a connected subgraph of G. The set of all

connected partitions of G forms a geometric sub-lattice of $\prod(V(G))$. It is called the *connected partition lattice* of G. It is denoted by $\prod^c(G)$. The unique minimal element $\hat{0}$ of $\prod^c(G)$ is the finest partition of $V(G)$ consisting of only singletons, (see e.g. [12]).

Let π be a connected partition of $V(G)$ and $\Lambda_i \cong G[\pi]$. The partition type of Λ_i is $\lambda(\Lambda_i) = \lambda(\pi)$. The *partition deck* of G is the set *partitions*$(G) = \{(\lambda, k_\lambda)\}$ where $\lambda \vdash v(G)$ and there are precisely k_λ connected partitions π of $V(G)$ such that $\lambda(\pi) = \lambda$.

2.3. **The folded connected partition lattice of a graph.**

The connected partition lattice of G may be *folded* by identifying those partitions π and σ for which $G[\pi]$ and $G[\sigma]$ are isomorphic. In other words, we would like to define a partial order on mutually non-isomorphic graphs induced by connected partitions. This is done as follows.

For arbitrary graphs H_1 and H_2, define $H_1 \preceq_\pi H_2$ if there is a family π of disjoint subsets of $V(H_2)$ such that $H_2[\pi] \cong H_1$.

Definition 2. Given a graph G, let $\mathcal{L}^c(G) = \{\Lambda_i;\ i \in [p]\}$ be a set of graphs such that

(1) the graphs $\Lambda_i;\ i \in [p]$ are mutually non-isomorphic,
(2) each Λ_i is isomorphic to a graph induced by a connected partition of $V(G)$,
(3) for each connected partition π of $V(G)$, there is a unique graph Λ_i isomorphic to $G[\pi]$,
(4) $c(\Lambda_i) \geq c(\Lambda_{i+1})$ for $i \in [p-1]$.

The poset $(\mathcal{L}^c(G), \preceq_\pi)$, along with edge labels $\#(\Lambda_i \xrightarrow{\pi} \Lambda_j)$ on all related pairs $\Lambda_i \preceq_\pi \Lambda_j$, is called the *folded connected partition lattice* of G. When there is no ambiguity, both the underlying set and the edge labelled lattice are denoted by $\mathcal{L}^c(G)$.

The partition lattice $\mathcal{L}^c(G)$ is called *fully labelled* if the graphs Λ_i are given. It is called *partially labelled* if the *type* of each Λ_i is specified as $\prod_j H_j^{k_j}$, where H_j are connected vertex induced subgraphs of G, but the isomorphism classes of H_j are not given. It is called *unlabelled (or abstract)* if Λ_i and their types are unknown.

Example 1. A graph G and its collection of distinct non-empty induced subgraphs along with their multiplicities are shown in Figure 1. The induced subgraph poset of G is shown in Figure 2. The isomorphism types of subgraphs of G induced by connected partitions of $V(G)$ and their multiplicities are shown in Figure 3. The folded connected partition lattices of G (labelled, partially labelled and abstract) are

shown in Figure 4. Note that the Hasse diagrams are only for illustration; they do not always display complete information since the posets under consideration also have edge labels. But in case of the induced subgraph poset, the edge labels on all related pairs can be computed from the edge labels shown in the Hasse diagram, (see Lemma 2.3 in [13]).

The graphs $K_{1,n}$ and K_2^n have the same folded connected partition lattice.

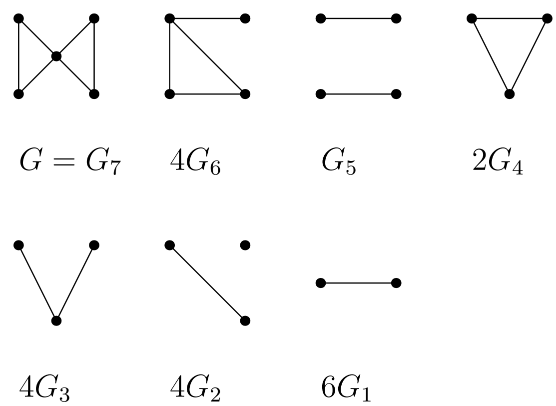


FIGURE 1. The non-empty induced subgraphs of G, and their multiplicities.

2.4. **The chromatic symmetric function and the symmetric Tutte polynomial.**

Definition 3. Let $x_i; i \in \mathbb{P}$ be commuting indeterminates. The *chromatic symmetric function* of a graph G is given by

$$X_G = \sum_{\kappa} \prod_{i} x_i^{|\kappa^{-1}(i)|} = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)} \tag{1}$$

where the summation is over all proper colourings $\kappa : V(G) \to \mathbb{P}$.

Stanley gave two non-isomorphic graphs on 5 vertices that have the same chromatic symmetric function. They are shown in Figure 5. Stanley reported that there are no non-isomorphic trees on 9 or fewer vertices having the same chromatic symmetric function.

Stanley also defined a symmetric polynomial generalisation of the Tutte polynomial.

Definition 4. Let $\kappa : V(G) \to \mathbb{P}$ be an arbitrary colouring (not necessarily a proper colouring) of G. Let $\beta(\kappa)$ denote the number of *bad*

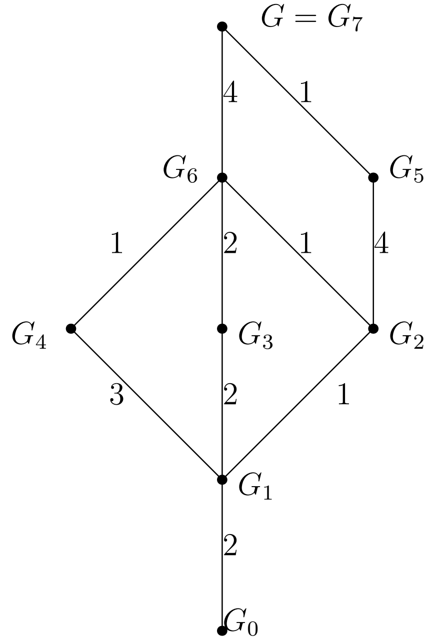


FIGURE 2. The induced subgraph poset of G.

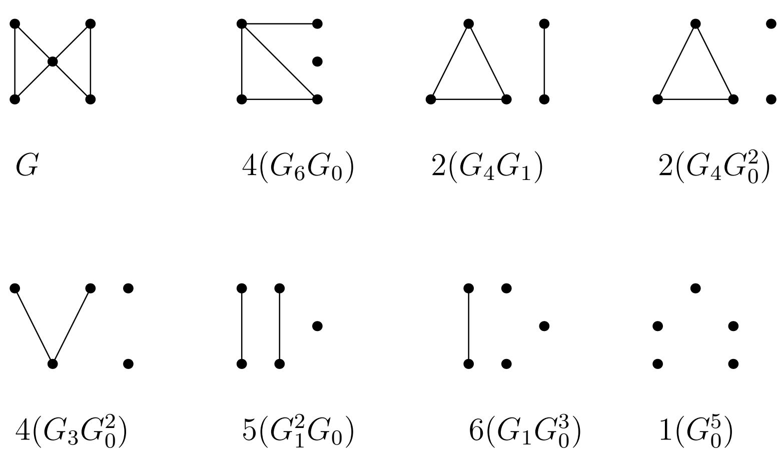


FIGURE 3. Graphs induced by the connected partitions of G, and their multiplicities.

edges in the colouring, that is, the edges that join vertices receiving identical colour. Let $x_i; i \in \mathbb{P}$ and t be commuting indeterminates.

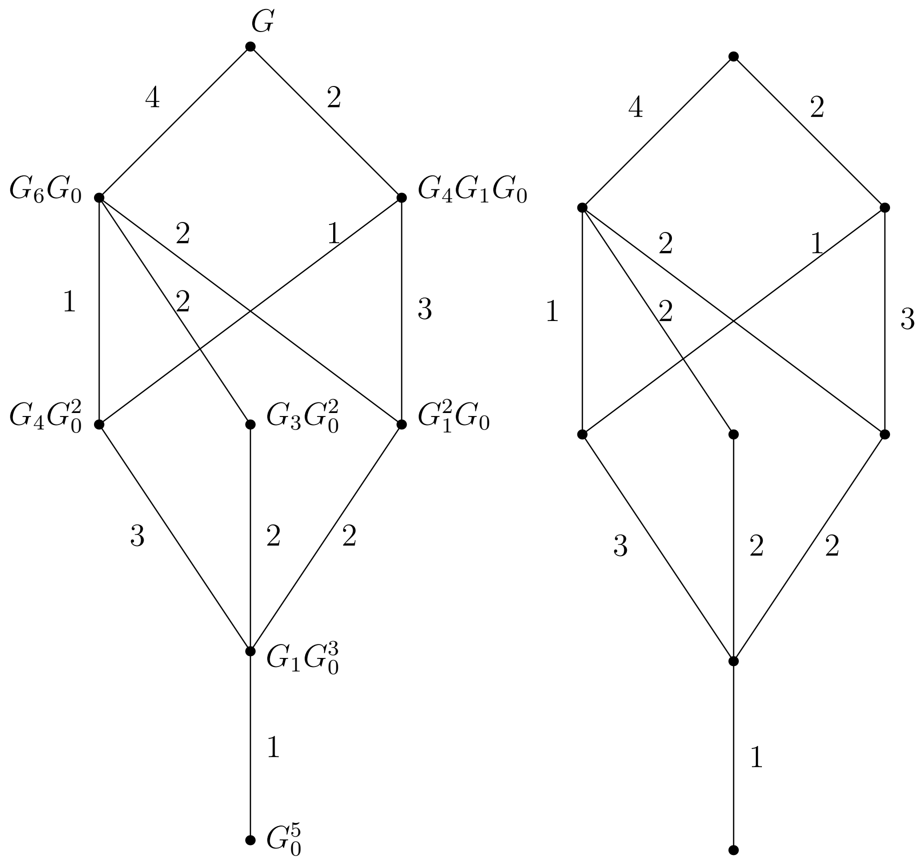


FIGURE 4. The folded lattice of connected partitions of G. It is called labelled if G_i are given (left), partially labelled if G_i are not given (left), and unlabelled (right).

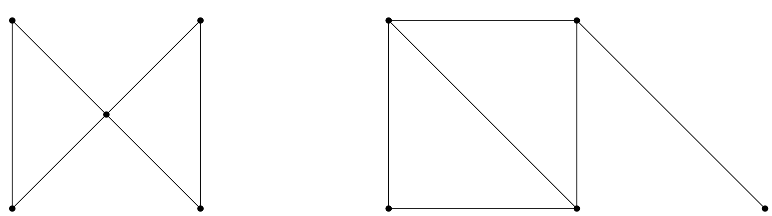


FIGURE 5. Non-isomorphic graphs having the same chromatic symmetric function

Then the *symmetric Tutte polynomial* of G is given by

$$X_G(t) = \sum_{\kappa} (1+t)^{\beta(\kappa)} \prod_i x_i^{|\kappa^{-1}(i)|} = \sum_{\kappa} (1+t)^{\beta(\kappa)} \prod_{v \in V(G)} x_{\kappa(v)} \qquad (2)$$

where the summation is over all colourings.

The symmetric Tutte polynomial specialises to Tutte polynomial as well as the chromatic symmetric function, and is equivalent to another invariant called the *weighted chromatic polynomial* for unit weights, denoted by $U(G)$. It was introduced by Noble and Welsh [9]. They asked if there are non-isomorphic trees having the same weighted chromatic polynomial with unit weights. Equivalently, they asked if the symmetric Tutte polynomial distinguishes trees. I will show that this question is equivalent to Stanley's original question for the chromatic symmetric function for trees, although, in general, $X_G(t)$ may be much stronger than X_G.

2.5. **The induced subgraph poset and Ulam's conjecture.**

The *vertex deck* of a graph G is the multi-set of unlabelled vertex deleted subgraphs of G. The *edge deck* of G is the multi-set of unlabelled edge deleted subgraphs of G. Two important conjectures in graph theory are the *vertex reconstruction conjecture* of Ulam and the *edge reconstruction conjecture* of Harary. Ulam's conjecture states that, provided $v(G) > 2$, G can be constructed up to isomorphism from its vertex deck. Harary's conjecture states that, provided $e(G) > 3$, G can be constructed up to isomorphism from its edge deck.

The following theorem summarises several results from [13]).

Theorem 1.

(1) *Ulam's conjecture is true if and only if all non-empty graphs can be constructed up to isomorphism from their induced subgraph posets.*
(2) *Ulam's conjecture is true if and only if the induced subgraph posets of all non-empty graphs have only trivial automorphisms.*
(3) *Every tree can be constructed up to isomorphism from its induced subgraph poset.*
(4) *The following invariants of a non-empty graph can be computed from its induced subgraph poset:*
    (a) *the number of spanning trees, (and hence whether the graph is connected),*
    (b) *the number of unicyclic graphs containing a cycle of a specified length, (and hence the number of hamiltonian cycles in the graph),*
    (c) *the number of spanning subgraphs having specified numbers of vertices and edges in their components,*
    (d) *the characteristic polynomial, the chromatic polynomial and the rank polynomial of the graph.*

The proof of Theorem 1(4) presented in [13] is difficult. The only fact from Theorem 1 that will be used in the constructions in Theorems 2 and 3 is that whether G is connected or not can be recognised from its induced subgraph poset $\mathcal{P}_v(G)$. The following lemmas proves this fact in a straight forward way. It is hoped that the idea of the following lemma and a variant of Kocay's lemma to be presented in the following section will significantly simplify the entire proof of Theorem 1. Details of such a proof will be made available in preprint form in future.

Lemma 1. *Whether a non-empty graph G is connected or not can be recognised from its induced subgraph poset $\mathcal{P}_v(G)$.*

Proof. A graph is connected if and only if at least two of its single vertex deleted subgraphs are connected. Therefore, the claim immediately follows by induction on the number of vertices. ☐

3. Constructions

The purpose of this section is to present constructions that relate the abstract induced subgraph poset and the abstract folded connected partition lattice. The constructions will be based on a lemma of Kocay [4] and Theorem 1.

3.1. Kocay's lemma and a variant.

Let $S = (F_1, F_2, \ldots, F_k)$ be a tuple of graphs. Let $cov(S \to H)$ denote the number of tuples $(H_1, H_2, \ldots, H_k)$ of subgraphs of H such that $H_i \cong F_i \,\forall\, i$, and $\cup_{i=1}^k H_i = H$. We call it the number of *covers* of H by S.

Lemma 2. (Kocay's Lemma [4]**)**

$$\prod_{i=1}^{k} \#(F_i \to G) = \sum_H cov(S \to H)\#(H \to G) \tag{3}$$

where the summation is over all isomorphism types of subgraphs of G. Also, if $v(F_i) < v(G) \,\forall\, i$ then $\sum_H cov(S \to H)\#(H \to G)$ over all isomorphism types H of spanning subgraphs of G can be reconstructed from the vertex deck of G.

Let $cov(S \xrightarrow{v} H)$ be the number of tuples $(X_1, X_2, \ldots, X_k)$ of subsets of $V(H)$ such that $H[X_i] \cong F_i \,\forall\, i$, and $\cup_{i=1}^k X_i = V(H)$. It is the number of *vertex covers* of H by S. The following lemma is a variant of Kocay's lemma.

Lemma 3.

$$\prod_{i=1}^{k} \#(F_i \xrightarrow{v} G) = \sum_{H} cov(S \xrightarrow{v} H)\#(H \xrightarrow{v} G) \tag{4}$$

where the summation is over all isomorphism types of induced subgraphs of G. If $v(F_i) < v(G)\,\forall\, i$ then $cov(S \xrightarrow{v} G)$ can be computed from the induced subgraph poset of G.

Proof. The LHS counts the number of tuples $(X_1, X_2, \dots, X_k)$ of subsets of $V(H)$ such that $H[X_i] \cong F_i\,\forall\, i$. The RHS is written by grouping the tuples in classes so that the tuples in each class cover the same vertex subset of $V(G)$. To prove the second part, we write Equation (4) in the following form:

$$cov(S \xrightarrow{v} G) = \prod_{i=1}^{k} \#(F_i \xrightarrow{v} G) - \sum_{H \subsetneq G} cov(S \xrightarrow{v} H)\#(H \xrightarrow{v} G)$$

where the summation on the RHS is over all isomorphism types of induced proper subgraphs of G. Each term $cov(S \xrightarrow{v} H)$ on the RHS can be expanded recursively on the induced subgraph poset by applying the above equation. □

To demonstrate the power of the above variant of Kocay's lemma we give two corollaries.

Corollary 1. *The chromatic symmetric function, and hence the chromatic polynomial, of a graph G can be computed from its induced subgraph poset $\mathcal{P}_v(G)$.*

Proof. We apply Lemma 3 for the tuple $S = (F_1, F_2, \dots, F_m)$, where $F_i \cong (K_1)^{k_i} \forall i$, and $\sum_i k_i = v(G)$. Since $\#(F_i \xrightarrow{v} G)$ is known for each i, (see Lemma 2.6 in [13]) we can compute $a_{k_1,k_2,\dots,k_v} = cov(S \xrightarrow{v} G)$, which is the number of proper colourings in which the colour i is used k_i times, from $\mathcal{P}_v(G)$. The chromatic polynomial is a specialisation of the chromatic symmetric function, see [11]. □

In fact we can easily compute the symmetric Tutte polynomial.

Corollary 2. *The symmetric Tutte polynomial $X_G(t)$ of a graph G can be computed from its induced subgraph poset $\mathcal{P}_v(G)$.*

Proof. We apply Lemma 3 for the tuple $S = (F_1, F_2, \dots, F_m)$, where F_i are induced subgraphs of G, and $\sum_i v(F_i) = v(G)$. Each ordered partition $(X_1, X_2, \dots, X_m)$ of $V(G)$ such that $G[X_i] \cong F_i; i \in [m]$ contributes a term $(1+t)^{\sum_{i\in[m]} e(F_i)} \prod_{i\in[m]} x_i^{v(F_i)}$ to the expression for

$X_G(t)$. Here $v(F_i)$ and $e(F_i)$ are known from the induced subgraph poset. Therefore,

$$X_G(t) = \sum_S cov(S \xrightarrow{v} G)(1+t)^{\sum_{i\in[m]} e(F_i)} \prod_{i\in[m]} x_i^{v(F_i)}.$$

We perform the summation over all *distinct* S, where we interpret the word 'distinct' as follows: we say $S = (F_1, F_2, \ldots, F_m)$ and $S' = (F'_1, F'_2, \ldots, F'_l)$ are identical if $(m = l)$ and $F_j \cong F'_j$ for all $j \in [m]$, else we say that S and S' are distinct. □

Yet another stronger invariant may be obtained. Consider a generalisation of proper colouring: suppose we want to assign each vertex u of a graph G on d vertices a nonempty subset $C(u)$ of $[d]$ of colours such that if two vertices p and q are adjacent in G then $C(p)$ and $C(q)$ are disjoint. We call it a *proper set colouring* of G. Let $a_{k_1,k_2,\ldots,k_d}$ be the number of ways of properly set colouring G so that colour i is used k_i times. When $k_i < d \forall i$, we apply Lemma 3 as in Corollary 1, without the restriction that $\sum_i k_i = d$, thus counting $a_{k_1,k_2,\ldots,k_d}$. If some of the k_i are equal to d then $a_{k_1,k_2,\ldots,k_d} = 0$ unless the graph is empty. A *set colouring symmetric function* could be defined in a similar way as the chromatic symmetric function. It is expected to contain a lot of information about the graph. This invariant or its generalisations may be interesting in their own right, but in this paper we do not go into them. It is conceivable that such generalisations are computable from the induced subgraph poset by applying Kocay's Lemma.

Other known proofs of the above results are fairly non-trivial See, for example, the induced subgraph expansion of the chromatic polynomial in [1], the reconstructibility of the chromatic symmetric function in [3], and also the reconstructibility of the chromatic polynomial in [2]. On the other hand the results presented above are stronger in the sense that they do not make use of the induced subgraphs. Also, as far as I am aware, the reconstructibility of the symmetric Tutte polynomial presented above is a new result.

3.2. **Relating the induced subgraph poset and the folded connected partition lattice.**

Theorem 2. *The partially labelled folded connected partition lattice $\mathcal{L}^c(G)$ of a graph G can be constructed from its induced subgraph poset $\mathcal{P}_v(G)$.*

Proof. Let the set of connected graphs in $\mathcal{P}_v(G)$ be $\mathcal{H} = \{H_i; 0 \leq i \leq k\}$: by Theorem 1, whether a graph $G_i \in \mathcal{P}_v G$ is connected or not is determined by its induced subgraph poset $\mathcal{P}_v(G_i)$. Now let $A = \prod_{i=1}^{a} A_i$ and $B = \prod_{i=1}^{b} B_i$ be graphs whose components are isomorphic to members of $\mathcal{H}$. We first show that $\#(A \xrightarrow{\pi} B)$ can be computed from $\mathcal{P}_v(G)$.

By Lemma 3, $\#(A \xrightarrow{\pi} H)$ can be computed from $\mathcal{P}_v(H)$ for any graph H. Let $\sigma : [a] \to [b]$ be an arbitrary map, and $\{X_i; i \in [m_\sigma]\}$ be the resulting partition of $[a]$, (that is, p and q are in the same cell if and only if $\sigma(p) = \sigma(q)$, and there are m_σ such cells). For each cell X_i of σ, define a graph $A_{X_i} = \prod_{j \in X_i} A_j$. We have

$$\#(A \xrightarrow{\pi} B) = \sum_{\sigma} \prod_{i=1}^{m_\sigma} \#(A_{X_i} \xrightarrow{\pi} B_{\sigma(X_i)})$$

where the summation is over *mutually inequivalent* maps in the following sense. Let σ and π be two maps from $[a]$ to $[b]$, and $\{X_i;\, i \in [m_\sigma]\}$ and $\{Y_i;\, i \in [m_\pi]\}$ be the corresponding partitions of $[a]$, then $\sigma \sim \pi$ if $m_\sigma = m_\pi$, $A_{X_i} \cong A_{Y_i}$ for all i, and $\sigma(X_i) = \pi(Y_i)$ for all i. The summation is performed by selecting one representative σ from each equivalence class of $\sim$. By Lemma 3, each factor $\#(A_{X_i} \xrightarrow{\pi} B_{\sigma(X_i)})$ can be computed.

To complete the construction of the partially labelled folded connected partition lattice of G, we perform the above computation for all pairs of graphs A and B for which $\#(A \xrightarrow{\pi} G)$ and $\#(B \xrightarrow{\pi} G)$ are non-zero and $v(A) = v(B) = v(G)$. $\square$

Theorem 3. *The induced subgraph poset $\mathcal{P}_v(G)$ of a graph G can be constructed from its partially labelled folded connected partition lattice $\mathcal{L}^c(G)$.*

Proof. Let $\mathcal{L}^c_G = \{\Lambda_i; i \in [p]\}$. Let $\mathcal{H} = \{H_i; 0 \leq i \leq k\}$ be the set of connected graphs in $\mathcal{P}_v(G)$. Since $\mathcal{L}^c_G$ is partially labelled, we suppose that the type of each Λ_i is given as $\prod_{j \in I} H_j^{k_j}$ for some $I \subset \{0, 1, \ldots, k\}$ and k_j are all positive. Recall that that "$\mathcal{L}^c_G$ is partially labelled" means that H_i themselves are not known up to isomorphism, but all we know is that there are distinct connected graphs *named* $H_i; 0 \leq i \leq k$ that

appear as components of members of $\mathcal{L}^c_G$. Thus the list of the *names* of connected induced subgraphs of G is obtained from the labels of Λ_i.)

Suppose that Λ_i are enumerated so that $c(\Lambda_i) \geq c(\Lambda_j)$ whenever $i < j$. Suppose the minimal element Λ_1 is labelled H_0^d. Therefore $v(G) = d$ is known from $\mathcal{L}^c_G$. Also, H_0 must be isomorphic to K_1.

For each graph H_i, there is a unique vertex of $\mathcal{L}^c_G$ that is labelled $H_i H_0^{d-v(H_i)}$. Thus $\#(H_0 \xrightarrow{v} H_i) = v(H_i)$ is known. So we may suppose that H_i are enumerated so that $v(H_i)$ are in non-decreasing order. Let $v(H_i) = d_i$. For any H_i and H_j, for $i, j > 0$, we have $\#(H_i \xrightarrow{v} H_j) = \#(H_i H_0^{d-d_i} \xrightarrow{\pi} H_j H_0^{d-d_j})$, therefore, $\#(H_i \xrightarrow{v} H_j)$ is known from $\mathcal{L}^c_G$.

To construct $\mathcal{P}_v(G) = \{G_i; 0 \leq i \leq n\}$, we must construct types of all induced subgraphs $G_i; i \in [n]$ of G, and compute $\#(G_i \xrightarrow{v} G_j)$ for all $i, j \in [n]$. This is done by induction on the number of vertices of G.

The base case, when $v(G) = 2$, is trivial: if the connected partition lattice has only one vertex then $G \cong (K_1)^2$, else $G \cong K_2$.

Let the result be true for all graphs having fewer than $v(G)$ vertices. First observe that $\mathcal{L}^c(H_i)$ is known for each H_i. It is the down-set of the vertex of $\mathcal{L}^c(G)$ that is labelled $H_0^{d-d_i} H_i$, except that the labels in $\mathcal{L}^c(H_i)$ are obtained by taking out the factor $H_0^{d-d_i}$ from the corresponding labels in $\mathcal{L}^c(G)$. Thus by induction hypothesis, $\mathcal{P}_v(H_i)$ is known for each H_i except possibly for H_k: if G is connected then it is labelled H_k, so $\mathcal{P}_v(H_k)$ is yet to be constructed. So we consider two cases.

Case 1: G is a disconnected graph. In this case $\mathcal{P}_v(H_i)$ is known for each H_i. Let $A = \prod_{i=1}^a A_i$ and $B = \prod_{i=1}^b B_i$ be graphs whose components are isomorphic to members of $\mathcal{H}$. We can argue as in Theorem 2 (replacing $\#(... \xrightarrow{\pi} ...)$ by $\#(... \xrightarrow{v} ...)$) to claim that $\#(A \xrightarrow{v} B)$ can be computed. The computation is performed first for $B = G$ and all graphs A, (so we know the list of distinct induced subgraphs of G), and then for all pairs of graphs A and B for which $\#(A \xrightarrow{v} G)$ and $\#(B \xrightarrow{v} G)$ are non-zero, and $v(A) \leq v(B) < d$.

Case 2: G is a connected graph. Suppose G is labelled H_k.

Let F be a graph on $d-1$ vertices. If F is connected then $\#(F \xrightarrow{v} G)$ has been counted above. Therefore, suppose that F is disconnected. If $\#(F \xrightarrow{v} G)$ is non-zero then there must be a Λ_i of the type $H_0 F$. Therefore, we consider only those Λ_i of the type $H_0^{k_0} \prod_{i \in I} H_i^{k_i}; I \subset [k-1]$, and count $\#(\Lambda_i / H_0 \xrightarrow{v} G)$, where Λ_i / H_0 denotes $H_0^{k_0 - 1} \prod_{i \in I} H_i^{k_i}$.

The following identity is used to count $\#(\Lambda_i/H_0 \xrightarrow{v} G)$ recursively.

$$\#(\Lambda_i/H_0 \xrightarrow{\pi} G) = \#(\Lambda_i/H_0 \xrightarrow{v} G) + \sum_j \#(\Lambda_i/H_0 \xrightarrow{\pi} \Lambda_j/H_0)\#(\Lambda_j/H_0 \xrightarrow{v} G) \tag{5}$$

where the summation is over j such that $j > i$ and Λ_j has at least one isolated vertex. The LHS is given by

$$\#(\Lambda_i/H_0 \xrightarrow{\pi} G) = k_0\#(\Lambda_i \xrightarrow{\pi} G) \tag{6}$$

The first factor in the summand is known from $\mathcal{L}^c(\Lambda_j)$, and the second factor in the summand is known if we use Equation (5) recursively starting from maximum j such that Λ_j has an isolated vertex. Thus we have computed $\#(\Lambda_i/H_0 \xrightarrow{v} G)$ for those Λ_i which contain an isolated vertex. By induction hypothesis $\mathcal{P}_v(\Lambda_i/H_0)$ is constructed for each of them.

If F is an arbitrary graph (with possibly fewer than $d-1$ vertices) then

$$(d - v(F))\#(F \xrightarrow{v} G) = \sum_{F'} \#(F \xrightarrow{v} F')\#(F' \xrightarrow{v} G),$$

where the summation is over all F' on $d-1$ vertices.

Therefore, now we know $\#(F \xrightarrow{v} G)$ for all graphs.

To construct the induced subgraph poset of G, the induced subgraph posets of Λ_i/H_0 are merged and top element G is added in the merged poset. Then for graphs F on fewer than $v(G)-1$ vertices, edge labels $\#(F \xrightarrow{v} G)$ computed above are assigned. This completes the construction of $\mathcal{P}_v(G)$ when G is connected.

The two steps above complete the induction step and the proof. □

Remark 1. Theorem 2 and Theorem 3 imply that given the abstract induced subgraph poset $\mathcal{P}_v(G)$, we can ultimately label each of its vertices as $\prod_i H_i^{k_i}$ where H_i are names of connected induced subgraphs.

Corollary 3. *If vertices of $\mathcal{L}^c(G)$ are labelled by their partition types then $\mathcal{P}_v(G)$ can be constructed.*

Proof. The problem is to construct a label of type $\prod_j H_j^{k_j}$ for each Λ_i, given the lattice $\mathcal{L}^c(G)$ labelled by partition types.

First, the minimal vertex of $\mathcal{L}^c(G)$ is labelled H_0^d, where d is the number of parts of its partition type.

Second, each vertex that is labelled $(a, 1, 1, \ldots, 1)$ is labelled $H_0^{d-d_i} H_i$, where $d - d_i$ is the number of 1's in its partition type. We can assume that $H_i; 0 \leq i \leq k$ are enumerated so that $v(H_i) = d_i$ are non-decreasing. At this point, $\#(H_j \xrightarrow{v} H_i)$ are known for all i, j, as in the proof of Theorem 3.

Third, the remaining vertices of $\mathcal{L}^c(G)$ are labelled in terms of H_i as follows. Let Λ_i be a vertex whose graph type has not been recognised yet. Let the type of Λ_i be $\prod_{j=0}^k H_j^{n_j}$, where some of the n_j's may be 0. Observe that $n_j = \#(H_j H_0^{d-d_j} \xrightarrow{\pi} \Lambda_i) - \sum_{i>j} \#(H_j \xrightarrow{v} H_i) n_i$ for all $j > 0$. Here the first term on the RHS counts $\#(H_j \xrightarrow{v} \Lambda_i)$, and the second term counts the number of induced subgraphs of Λ_i that are isomorphic to H_j and that appear as induced subgraphs of bigger components of Λ_i. This equation can be recursively solved starting from n_k. □

Theorem 4.

(1) *The induced subgraph poset $\mathcal{P}_v(G)$ of a graph G that has no isolated vertices can be constructed from its unlabelled folded connected partition lattice $\mathcal{L}^c(G)$, provided G is not isomorphic to $K_{1,n}$; $n > 1$ or $(K_2)^n$; $n > 1$.*

(2) *Ulam's conjecture is true if and only if all graphs other than $K_{1,n}$; $n > 1$ and $(K_2)^n$; $n > 1$ can be uniquely constructed from their unlabelled folded connected partition lattices.*

(3) *A tree or forest that is not isomorphic to $K_{1,n}$; $n > 1$ or $(K_2)^n$; $n > 1$ can be constructed up to isomorphism from its abstract folded connected partition lattice.*

Proof. It is clear that $\mathcal{L}^c((K_2)^n) \cong \mathcal{L}^c(K_{1,n})$ for each $n > 1$: both are just linear orders. Whether G is one of $K_{1,n}$ and $(K_2)^n$ for some $n > 1$ can be recognised from $\mathcal{L}^c(G)$, since no other graph G has $\mathcal{L}^c(G)$ that is a total order. Also, the abstract connected partition lattice is unchanged when isolated vertices are added in the graph. So we assume that G has no isolated vertices, and is different from $K_{1,n}$ and $(K_2)^n$ for any $n > 1$.

The theorem is proved by first constructing a partial labelling of $\mathcal{L}^c(G)$ in a unique way, and applying Theorem 3, or constructing G itself up to isomorphism from $\mathcal{L}^c(G)$.

Let $r : \mathcal{L}^c(G) \rightarrow N$ be a rank function on $\mathcal{L}^c(G)$ such that the rank of the minimal element is 0. Therefore, $r(\Lambda_i) = v(G) - c(\Lambda_i)$.

The partial labelling of $\mathcal{L}^c(G)$ is obtained by labelling its vertices in a bottom-up manner. The minimal element is labelled H_0^d, where d is unknown at this point. In general, we will first construct a labelling of

Λ_i modulo the number of isolated vertices in it, and then in the end recognise $v(G)$ from the number of non-empty components of G and $r(G)$.

Since both $\mathcal{L}^c(K_{1,2})$ and $\mathcal{L}^c((K_2)^2)$ are isomorphic (when unlabelled), it is not immediately obvious which vertex of $\mathcal{L}^c(G)$ corresponds to the π-subgraph $K_{1,2}$ and which one corresponds to the π-subgraph $(K_2)^2$. But the vertices corresponding to π-subgraphs $K_{1,2}$ and $(K_2)^2$ are unambiguously recognised if G has any of the graphs B_1, C_4, K_3K_2, $K_{1,3}K_2$, P_4K_2, $(K_{1,2})^2$, $K_{1,2}(K_2)^2$ and K_4 as a π-subgraph. The graph B_1 is shown in Figure 6. This is proved by constructing the abstract connected partition lattice for each of them and possibly other small graphs, and verifying two facts

(1) each graph in the above list is uniquely determined by its abstract folded connected partition lattice.
(2) the abstract folded connected partition lattice of each graph in the list has only the trivial automorphism.

For example, suppose C_4 is a π-subgraph of G. Since no other graph has the same abstract connected partition lattice as C_4, the vertex of $\mathcal{L}^c(G)$ whose down-set is isomorphic to $\mathcal{L}^c(C_4)$ must be labelled C_4 (modulo isolated vertices). Suppose a vertex Λ_i is covered by the vertex that is labelled C_4. If $\#(\Lambda_i \xrightarrow{\pi} C_4) = 2$ then $\Lambda_i \cong (K_2)^2$ (modulo isolated vertices), else $\#(\Lambda_i \xrightarrow{\pi} C_4) = 4$, in which case $\Lambda_i \cong (K_{1,2})$ (modulo isolated vertices).

If none of the above graphs is a π-subgraph of G then G is isomorphic to K_4/e or P_5 (a path on 5 vertices) or a 5-cycle (or $K_{1,n}$; $n > 1$ or $(K_2)^n$; $n > 1$, but we have assumed that G is not one them). If the graph G is isomorphic to K_4/e or P_5 or a 5-cycle then one can observe that G is uniquely determined by the abstract $\mathcal{L}^c(G)$, although $\mathcal{L}^c(G)$ has non-trivial automorphisms. The non-trivial automorphisms imply that although G is uniquely determined from $\mathcal{L}^c(G)$ in this case, all vertices of $\mathcal{L}^c(G)$ cannot be unambiguously labelled. Therefore, now we assume that G contains one of the graphs B_1, C_4, K_3K_2, $K_{1,3}K_2$, P_4K_2, $(K_{1,2})^2$, $K_{1,2}(K_2)^2$ and K_4 as a π-subgraph.

We recognise those Λ_i that are of the type $H_jH_0^{d-d_j}$ where H_j is a connected graph and $v(H_j) = d_j$. This is done by induction on the rank. The base case is $r(\Lambda_i) = 2$: both rank 2 graphs are recognised, and they are $K_{1,2}$ and $(K_2)^2$. Suppose all such vertices with rank at most k have been recognised. Let $r(\Lambda_i) = k + 1$. We use the fact that a graph H is connected if and only if at least 2 of its vertex deleted subgraphs are connected. Suppose that Λ_i covers at least two π-subgraphs of rank k having exactly one non-trivial component. Then

either Λ_i is of the type $H_j H_0^{d-d_j}$ or Λ_i is of the type $H_j K_2 H_0^{d-d_j-2}$. The necessary and sufficient condition for the latter case is that there is a π-subgraph $\Lambda_j = H_j H_0^{d-d_j}$ such that $e(\Lambda_j) = e(\Lambda_i) - 1$, $\#(\Lambda_j \xrightarrow{\pi} \Lambda_i) \geq 1$ and $\#((K_2)^2 \xrightarrow{\pi} \Lambda_i) = \#((K_2)^2 \xrightarrow{\pi} \Lambda_j) + e(\Lambda_j)$. Here the first two conditions imply that exactly one edge is deleted from Λ_i to get Λ_j, and the last condition implies that the deleted edge is not adjacent to any other edge in Λ_j. These conditions are recognised from $\mathcal{L}^c(G)$ and the induction hypothesis. Note that d_j is not known at this point.

The labels of the remaining Λ_i are recognised by a procedure exactly similar to the one in Corollary 3. In fact, in Corollary 3, only the labels of vertices that were labelled $(a, 1, 1, \ldots, 1)$ were used.

The second part follows from Theorem 3(1) and Theorem 1(1).

The third part of the corollary now follows from Theorem 3 and Theorem 1(3). □

3.3. **The Möbius function.** Let μ be the Möbius function of the unfolded connected partition lattice $\prod^c(G)$ of G. Let π and σ be any connected partitions of $V(G)$. If $G[\pi] \cong G[\sigma]$ then $\mu(\hat{0}, \pi) = \mu(\hat{0}, \sigma)$. Therefore, for any graph Λ_i in $\mathcal{L}^c(G)$, we define $\mu(\hat{0}, \Lambda_i) = \mu(\hat{0}, \pi)$ where π is some connected partition of $V(G)$ such that $G[\pi] \cong \Lambda_i$.

Lemma 4. *For any graph Λ_i in $\mathcal{L}^c(G)$, other than the graph induced by the finest partition, $\mu(\hat{0}, \Lambda_i)$ can be computed from $\mathcal{L}^c(G)$.*

Proof. We have

$$\sum_{\Lambda_k | \hat{0} \preceq_\pi \Lambda_k \preceq_\pi \Lambda_i} \mu(\hat{0}, \Lambda_k) \#(\Lambda_k \xrightarrow{\pi} \Lambda_i) = 0 \tag{7}$$

Thus if $\mu(\hat{0}, \Lambda_k)$ are known for all $\Lambda_k \prec_\pi \Lambda_i$ then $\mu(\hat{0}, \Lambda_i)$ can be computed. Therefore, we compute $\mu(\hat{0}, \Lambda_i)$ by solving Equation (7) recursively. □

Now we have another way of computing the chromatic symmetric function based on the expansion given by Stanley [11].

Corollary 4. *The chromatic symmetric function of a graph G without isolated vertices can be computed from its unlabelled folded connected partition lattice $\mathcal{L}^c(G)$, except when G is one of $K_{1,n}$; $n > 1$ or $(K_2)^n$; $n > 1$*

Proof.

$$X_G = \sum_{\Lambda_i \in \mathcal{L}^c(G)} \#(\Lambda_i \xrightarrow{\pi} G) \mu(\hat{0}, \Lambda_i) p_{\lambda(\Lambda_i)} \tag{8}$$

where $p_{\lambda(\Lambda_i)}$ is the power sum symmetric function given by

$$p_{\lambda(\Lambda_i)} = \prod_{k=1}^{l} \sum_i x_i^{\lambda_k}$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ is the partition type $\lambda(\Lambda_i)$ of Λ_i The vertex labels in $\mathcal{L}^c(G)$ do not have to be given since, by Theorem 4, either the labels or G can be constructed from $\mathcal{L}^c(G)$. □

Remark 2. It is not clear if there is a nice definition of $\mu(\Lambda_i, \Lambda_j)$ in general because the unfolded connected partition lattice $\prod^c(\Lambda_j)$ may contain partitions π and σ such that $\Lambda_j[\pi] = \Lambda_j[\sigma]$ but there may not be an automorphism of $\prod^c(\Lambda_j)$ that maps π to σ. Understanding the structure of the unfolded lattice may be quite relevant to Ulam's conjecture, and it would be interesting to investigate if the abstract unfolded connected partition lattice can be constructed from the folded connected partition lattice. Stanley's expansion of the chromatic symmetric function makes use of only $\mu(\hat{0}, \pi)$. The invariants that have $\mu(\pi, \sigma)$ in their expansions may be more difficult to reconstruct.

4. Trees and their symmetric polynomials

In this section I will present constructions that are valid only on trees. Let T be a tree, $\mathcal{P}_v(T)$ its induced subgraph poset, and $\mathcal{L}^c(T)$ its partially labelled folded connected partition lattice. Let $\mathcal{L}^c(T) = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_p\}$ where Λ_i are enumerated so that the number of components $c(\Lambda_i)$ are non-increasing. Let μ be the Möbius function of $\prod^c(T)$, and $\mu(\hat{0}, \Lambda_i)$ as defined earlier.

Lemma 5. *Whether a graph G is a tree can be recognised from its chromatic symmetric function X_G, and also from its partition deck partitions(G).*

Proof. The number of vertices is d if there is a term $x_1x_2\ldots x_d$ in X_G, or if there is a partition $(1, 1, \ldots, 1)$, with d 1's, in *partitions*(G). The number of edges is obtained from (λ, k_λ) in *partitions*(G), where λ is the partition $(2, 1, 1, \ldots, 1)$ with $d-2$ 1's. From X_G, the number of edges is obtained by looking at the coefficient of $x_1^2x_2x_3\ldots x_{d-1}$. If the coefficient is $\binom{d}{2} - m$ then $e(G) = m$. The number of components in G is obtained from *partitions*(G) by looking at the partition of smallest length. If $v(G) = d$ and $e(G) = d-1$ then G is tree if and only if it has no cycles. Therefore, G with d vertices and $d-1$ edges is a tree if and only if the number of acyclic orientations of G is 2^{d-1}. The number of

acyclic orientations can be calculated from the chromatic polynomial (see [10]), which in turn can be calculated from X_G. □

Lemma 6. *For any Λ_i in $\mathcal{L}^c(T)$,*

$$\mu(\hat{0}, \Lambda_i) = (-1)^{v(T)-c(\Lambda_i)} = (-1)^{r(\Lambda_i)} \tag{9}$$

where r is the rank function on $\mathcal{L}^c(T)$ with $r(\Lambda_1) = 0$.

Proof. The claim follows from the fact that the unfolded connected partition lattice of T is isomorphic to the power set lattice of $E(T)$. □

Lemma 7. *The partition deck of a tree T can be constructed from its chromatic symmetric function X_T.*

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash v(T)$. Let c_λ be the coefficient of $\prod_{i=1}^{l} x_i^{\lambda_i}$ in X_T. Let k_λ be the number of connected partitions π of $V(T)$ such that $\lambda(G[\pi]) = \lambda$. The numbers c_λ and k_λ satisfy

$$c_\lambda = \sum_{\lambda' \preceq \lambda} (-1)^{v(T)-l(\lambda')} k_{\lambda'} a(\lambda', \lambda) \tag{10}$$

where λ' are refinements of λ and $a(\lambda', \lambda)$ is the contribution to c_λ from $p_{\lambda'}$, that is, it is the coefficient of $\prod_{i=1}^{l} x_i^{\lambda_i}$ in $p_{\lambda'}$. The equation (10) can be recursively solved for k_λ. Thus we can construct the partition deck of $\mathcal{L}^c(T)$. □

Remark 3. The pair of graphs given by Stanley (Figure 5) do not have the same partition deck.

Lemma 8. *The chromatic symmetric function X_T of a tree T can be computed from the partition deck partitions(T).*

Proof. The chromatic symmetric function is given by

$$X_T = \sum k_\lambda (-1)^{v(T)-l(\lambda)} p_\lambda \tag{11}$$

where the sum is over all pairs (λ, k_λ) in *partitions*(T). □

Remark 4. Thus, in the case of a tree, the chromatic symmetric function does not contain more information than the partition deck.

Lemma 9. *The symmetric Tutte polynomial $X_T(t)$ of a tree T can be computed from its chromatic symmetric function X_T or from the partition deck partitions(T).*

Proof. It is somewhat easier to compute $X_T(t)$ in more detail. Let $\kappa : V(T) \to [r]$ be an arbitrary colouring of the tree T with exactly r colours. Let $(V_1, V_2, \ldots, V_r)$ be the partition of $V(T)$ in colour classes. For $i \in [r]$, let q_i be the number of edges having both ends in V_i, and

let $k_i = |V_i|$. Thus, associated with each colouring κ, we have vectors $\bar{q} = (q_1, q_2, \ldots, q_r)$ and $\bar{k} = (k_1, k_2, \ldots, k_r)$. Let $f(\bar{q}, \bar{k})$ be the number of colourings with associated vectors $\bar{q}$ and $\bar{k}$. We will prove the lemma by computing $f(\bar{q}, \bar{k})$ for each pair of vectors $\bar{q}$ and $\bar{k}$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ be a partition of multiplicity k_λ in the partition deck of T. Let $\bar{p}$ be a vector of length r having non-negative entries. Let $g(\bar{p}, \bar{k}, \lambda)$ be the number of set partitions $(A_1, A_2, \ldots, A_r)$ of $\{1, 2, \ldots, s\}$ such that, for each $i \in [r]$, $\sum_{j \in A_i} \lambda_j = k_i$ and $\sum_{j \in A_i} (\lambda_j - 1) = p_i$. We have the following recursion relating $g(\bar{p}, \bar{k}, \lambda)$ and $f(\bar{q}, \bar{k})$.

$$f(\bar{p}, \bar{k}) = \sum_{(\lambda, k_\lambda) \in partitions(T)} g(\bar{p}, \bar{k}, \lambda) k_\lambda - \sum_{\bar{q} > \bar{p}} f(\bar{q}, \bar{k}) \prod_{i=1}^{r} \binom{q_i}{p_i} \tag{12}$$

where $\bar{q} \geq \bar{p}$ denotes that the vector $\bar{q}$ dominates the vector $\bar{p}$ termwise, that is, $q_i \geq p_i$ for all $i \in [r]$, and $\bar{q} > \bar{p}$ means that $\bar{q} \geq \bar{p}$ and for at least one value of $i \in [r]$, $q_i > p_i$. This is explained as follows. Let $(U_1, U_2, \ldots, U_s)$ be a connected partition of $V(T)$ such that $|U_i| = \lambda_i$ for all $i \in [s]$. Let$(A_1, A_2, \ldots, A_r)$ a partition of $\{1, 2, \ldots, s\}$. Suppose $V_i = \cup_{j \in A_i} U_j$ is the i-th colour class of a colouring, for $i \in [r]$, and suppose that $|V_i| = k_i$ for each $i \in [r]$. Then there are at least $\sum_{j \in A_i} (\lambda_j - 1)$ bad edges in colour class V_i, for $i \in [r]$. Thus the partition $(V_1, V_2, \ldots, V_r)$ contributes to $f(\bar{p}, \bar{k})$ only if each V_i has exactly $\sum_{j \in A_i} (\lambda_j - 1) = p_i$ edges in it. Otherwise, the colouring with colour partition $(V_1, V_2, \ldots, V_r)$ had been accounted for in $f(\bar{q}, \bar{k})$ for some $\bar{q} > \bar{p}$. Thus the second summation in Equation (12) are precisely the *false positives* in the first summation. Equation (12) can be solved recursively for $f(\bar{q}, \bar{k})$ in the top-down order on the vectors $\bar{q}$ starting from $\bar{q} = \bar{k} - (1)_r$, where $(1)_r$ is the vector of r 1's. When $\bar{q} = \bar{k} - (1)_r$, the quantity $f(\bar{q}, \bar{k})$ is directly obtained from the connected partitions of the type $(k_1, k_2, \ldots, k_r)$. The above computation is repeated for each vector $\bar{k}$ for which $\sum_i k_i = v(T)$.

Now $X_T(t)$ can be computed from $f(\bar{q}, \bar{k})$: each pair of vectors $(\bar{q}, \bar{k})$ contributes a term

$$f(\bar{q}, \bar{k}) t^{\sum_i q_i} \prod_i x_i^{k_i}$$

to $X_T(t)$. Since $f(\bar{q}, \bar{k})$ was computed from $partitions(T)$, from Lemma 7 it follows that $X_T(t)$ can be computed from X_T. $\square$

The following lemma only summarises a part of the above proof since it will be useful later.

Lemma 10. *Let T be a tree. Let $\bar{k}$ and $\bar{q}$ be arbitrary vectors, both of length r, such that $\bar{k}$ has positive entries and $\bar{q}$ has non-negative entries. Then the number $f(\bar{q}, \bar{k})$ of ordered partitions $(V_1, V_2, \ldots, V_r)$ of $V(T)$ such that $|V_i| = k_i$ and $e(T[V_i]) = q_i$ for all i can be computed from the partition deck partitions(T) of T or from the chromatic symmetric function X_T of T.*

Remark 5. Since the chromatic symmetric function is a specialisation of the symmetric Tutte polynomial, Lemma 9 implies that Stanley's question (whether the chromatic symmetric function distinguishes trees) and the question of Noble and Welsh (whether their weighted chromatic function with unit weights, which is equivalent to the symmetric Tutte polynomial, distinguishes trees) are equivalent.

The partition deck and numbers $f(\bar{q}, \bar{k})$ are perhaps sufficient to construct the tree, and answer Stanley's question in the affirmative. Here we demonstrate a simple application of Lemma 9.

Let the *degree of a subtree* of a tree T be the number of edges in T with one end in the subtree and one end outside the subtree.

Lemma 11. *From the chromatic symmetric function of a tree, the number of its subtrees with a given number of vertices and a given degree can be computed. In particular, the chromatic symmetric function of a tree determines its degree sequence.*

Proof. Let $\bar{v} = (k, v(T) - k)$ and $\bar{q} = (k-1, v(T) - k - d)$. Now the number $f(\bar{v}, \bar{q})$ counts the number of subtrees on k vertices having degree d. Setting $k = 1$, we get the degree sequence. ☐

5. Edge reconstruction and the edge subgraph poset

Here we will define the notion of the edge subgraph poset of a graph analogous to the induced subgraph poset, and develop edge reconstruction theory for it. It turns out that an equivalence between edge reconstruction conjecture and reconstruction from the edge subgraph poset is not as obvious as our earlier result about Ulam's conjecture.

Definition 5 (The edge-subgraph poset of a graph)**.** Given a graph G, let $\mathcal{P}_e(G) = \{G^i; i \in [m]\}$ be a set of graphs such that

(1) the graphs $G^i; i \in [m]$ are non-empty,
(2) the graphs $G^i; i \in [m]$ mutually non-isomorphic,
(3) every G^i is isomorphic to an edge subgraph of G, and every edge subgraph of G having a non-empty edge set is isomorphic to a unique G^i,
(4) $e(G^i) \leq e(G^{i+1})$ for $i \in [m]$, so $G^1 \cong K_2$.

Let $\preceq_e$ be a partial order relation on $\mathcal{P}_e(G)$ defined by $G^i \preceq_e G^j$ if G^i is isomorphic to an edge-subgraph of G^j. The partially ordered set $(\mathcal{P}_e(G), \preceq_e)$ is *edge labelled* by associating with each related pair $G^i \preceq_e G^j$ of graphs the number $\#(G^i \xrightarrow{e} G^j)$. The edge labelled poset itself is also denoted by $\mathcal{P}_e(G)$, and will be simply referred to as the *edge subgraph poset* of G.

We say that two edge subgraph posets are isomorphic if they are isomorphic as posets, and there is an isomorphism between them that preserves the edge labels. Formally, $\mathcal{P}_e(G)$ is isomorphic to $\mathcal{P}_e(H)$ if there is a bijection π from $\mathcal{P}_e(G)$ to $\mathcal{P}_e(H)$ such that $\#(G^i \xrightarrow{e} G^j) = \#(\pi(G^i) \xrightarrow{e} \pi(G^j))$ for all G^i and G^j in $\mathcal{P}_e(G)$. The *unlabelled or abstract* edge subgraph poset of G is the isomorphism class of $\mathcal{P}_e(G)$. Unless specified otherwise, we will always assume an edge subgraph poset to be unlabelled. An isomorphism from an edge subgraph poset to itself is called an automorphism of the edge subgraph poset.

Since addition of isolated vertices to a graph does not change the edge subgraph poset, we assume unless stated otherwise that graphs considered below are without isolated vertices.

Recall that the proof of Theorem 1 (1) in [13] required induction on the number of vertices. A similar proof by induction on the number of edges is not immediately obvious for the edge version because for any $n > 1$ the graphs $K_{1,n}$ and $(K_2)^n$ are non-isomorphic but their edge subgraph posets are isomorphic. Thus we already have infinitely many graphs without isolated vertices that cannot be constructed from their edge subgraph posets. Also, the graphs that are not edge reconstructible cannot be constructed from their edge subgraph posets.

A counter example to edge reconstruction is a maximal set of mutually non-isomorphic graphs having the same collection of edge subgraphs. The two known counter examples to edge reconstruction are $\{K_{1,2}, (K_2)^2\}$ and $\{K_{1,3}, K_3\}$. Let $\mathcal{C}$ be the class of counter examples to edge reconstruction. A counter example to $\mathcal{P}_e$-reconstruction is a maximal set of mutually non-isomorphic graphs having the same edge subgraph poset. Let $\mathcal{C}_p$ be the class of counter examples to $\mathcal{P}_e$-reconstruction. The counter examples to edge reconstruction are also counter examples to $\mathcal{P}_e$-reconstruction. Therefore, $\mathcal{C} \subseteq \mathcal{C}_p$. We are interested in characterising counter examples to $\mathcal{P}_e$-reconstruction that are not counter examples to edge reconstruction.

We will sometimes abuse the notation slightly and say that $\{G_1, G_2\}$ is a counter example to edge reconstruction (or $\mathcal{P}_e$-reconstruction) even if it is likely that there are graphs $G_3, G_4 \ldots$ non-isomorphic to G_1 and G_2 having the same collection of edge subgraphs (or the same edge

subgraph poset) as G_1 and G_2. Note that the counter examples defined above really contain isomorphism types although we will sometimes talk of a labelled graph belonging to or not belonging to one of the classes.

Characterising counter examples in $\mathcal{C}_p/\mathcal{C}$ requires enumerating all counter examples on at most 6 edges because only when the number of edges is at least 7, the counter examples have a nice structure. So we first enumerate counter examples on at most 7 edges, and then do induction on the number of edges.

Let T_4, T_5, B_1, B_2, B_3 and B_4 be graphs shown in Figure 6.

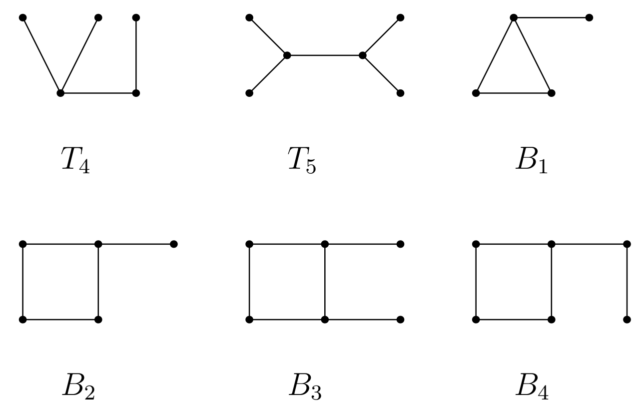


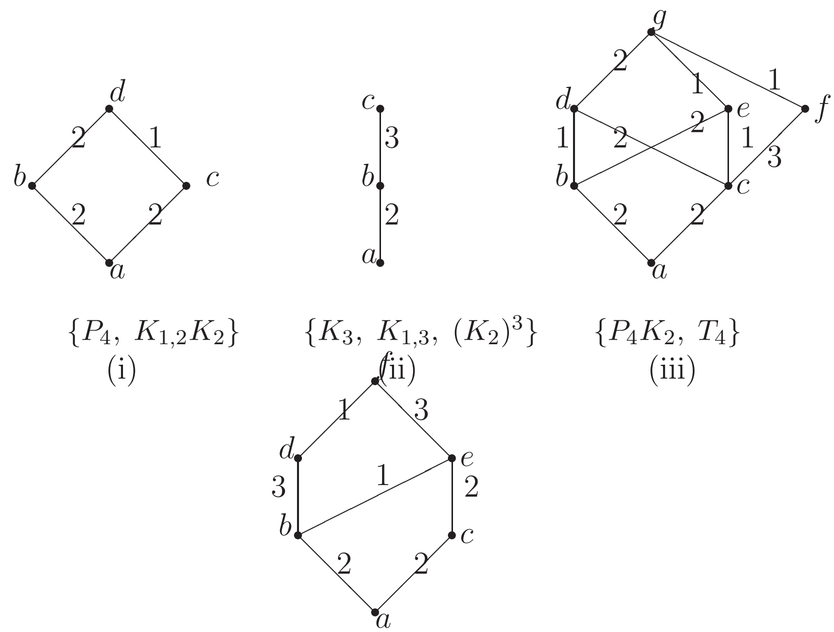
FIGURE 6. Some graphs

Lemma 12. *The only counter examples to $\mathcal{P}_e$-reconstruction on at most four edges are:* $\{K_{1,2},\ (K_2)^2\}$, $\{K_3,\ K_{1,3},\ (K_2)^3\}$, $\{P_4\ ,\ K_{1,2}K_2\}$, $\{K_{1,4},\ (K_2)^4\}$, $\{P_4K_2,\ T_4\}$, $\{C_4,\ (K_{1,2})^2\}$ *and* $\{K_3K_2,\ K_{1,3}K_2\}$

Proof. The lemma is proved by constructing edge subgraph posets of all graphs on at most four edges, and verifying that the graphs in each class listed above have isomorphic edge subgraph posets. $\square$

Although it is not difficult to list the edge subgraph posets, we do list some of them below to illustrate certain ideas that are useful later.

In the example in Figure 7(iii), one can observe that the down-set of f is isomorphic to the poset in (ii). But since none of the graphs P_4K_2 and T_4 has a triangle, and these are the only graphs whose edge subgraph poset is as shown in (iii), we can conclude that whenever (iii) appears as a sub-poset of an edge subgraph poset, the point f must be either $K_{1,3}$ or $(K_2)^3$. In (iv), the point d is K_3 or $K_{1,3}$ depending on whether the graph is K_3K_2 or $K_{1,3}K_2$, respectively. But in (iv), the edge subgraph corresponding to e is uniquely determined. It must be $K_{1,2}K_2$ although



$\{P_4,\ K_{1,2}K_2\}$ (i) $\{K_3,\ K_{1,3},\ (K_2)^3\}$ (ii) $\{P_4K_2,\ T_4\}$ (iii)

$\{K_3K_2,\ K_{1,3}K_2\}$ (iv)

FIGURE 7. Counter examples to $\mathcal{P}_e$-reconstruction

the down-set of e is the poset in (i). This is because P_4 is not a subgraph of any of K_3K_2 or $K_{1,3}K_2$. Once e is recognised to be $K_{1,2}K_2$, points b and c are uniquely recognised as $K_{1,2}$ and $(K_2)^2$, respectively. This is because the down-set of e has no nontrivial automorphisms. So the edge deleted subgraph that appears twice is $(K_2)^2$, and the one that appears once is $K_{1,2}$. Such arguments will be frequently used to recognise some of the subgraphs in the edge subgraph poset that is unlabelled otherwise.

In Figure 8 one can observe that P_5 is recognised because no other edge subgraph poset is isomorphic to $\mathcal{P}_e(P_5)$. But none of its subgraphs are recognised since there is a non-trivial automorphism of $\mathcal{P}_e(P_5)$ which interchanges b & c, and d & e. But we can claim that if d is labelled P_4 then b must be labelled $(K_2)^2$. The edge subgraph poset of B_1 has an automorphism that interchanges points d and e. Therefore, one of them is a K_3 and the other is $K_{1,3}$. Other subgraphs are uniquely recognised: f must be P_4, b must be $K_{1,2}$ and c must be $(K_2)^2$. The edge subgraph poset of $K_{1,2}(K_2)^2$ has no nontrivial automorphisms, and there is no other edge subgraph poset that

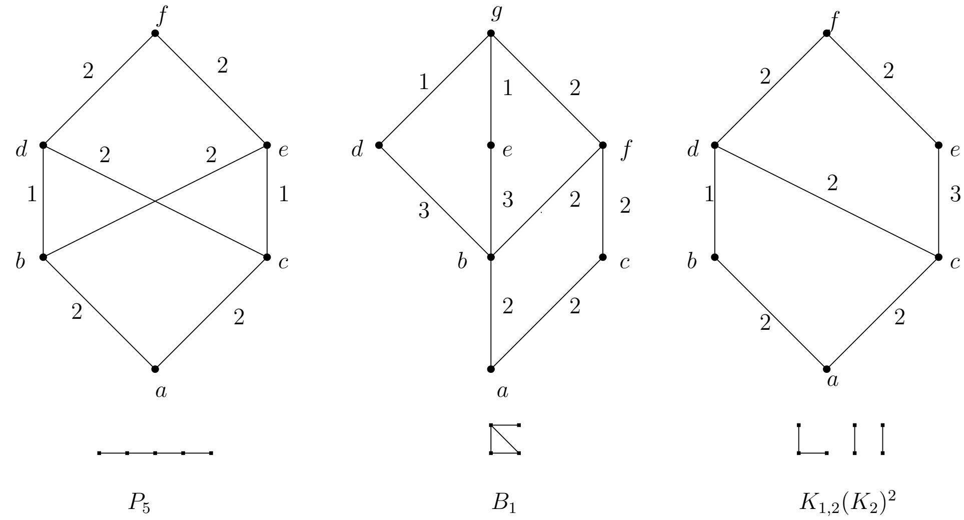


FIGURE 8. Not counter examples to $\mathcal{P}_e$-reconstruction

is isomorphic to $\mathcal{P}_e(K_{1,2}(K_2)^2)$. Therefore, $\mathcal{P}_e(K_{1,2}(K_2)^2)$ can be fully labelled as $a = K_2$, $b = K_{1,2}$, $c = (K_2)^2$, $d = K_{1,2}K_2$, $e = (K_2)^3$ and $f = K_{1,2}(K_2)^2$. This information is very useful because whenever $K_{1,2}(K_2)^2$ appears as a subgraph, all its subgraphs are uniquely labelled.

Lemma 13. *The only counter examples to $\mathcal{P}_e$-reconstruction on five edges are:* $\{K_{1,5},\ (K_2)^5\}$, $\{K_3(K_2)^2,\ K_{1,3}(K_2)^2\}$, $\{K_3K_{1,2},\ K_{1,3}K_{1,2}\}$, $\{C_4K_2,\ T_5\}$ *and* $\{P_6,\ B_2\}$*,*

Lemma 14. *The only counter examples to $\mathcal{P}_e$-reconstruction on six edges are:* $\{K_{1,6},\ (K_2)^6\}$, $\{K_3(K_2)^3,\ K_{1,3}(K_2)^3\}$, $\{(K_3)^2,\ (K_{1,3})^2\}$, $\{K_3P_4,\ K_{1,3}P_4\}$, $\{K_3K_{1,2}K_2,\ K_{1,3}K_{1,2}K_2\}$ *and* $\{B_3,\ B_4\}$.

Lemma 15. *The only counter examples to $\mathcal{P}_e$-reconstruction on seven edges are:* $\{K_{1,7},\ (K_2)^7\}$, $\{(K_3)^2K_2,\ (K_{1,3})^2K_2\}$ *and* $\{K_3F, K_{1,3}F\}$, *where F is any of the graphs P_5, C_4, B_1, $(K_{1,2})^2$, P_4K_2, $K_{1,2}(K_2)^2$ and $(K_2)^4$.*

The proofs of Lemmas 13, 14 and 15 are skipped at this point. They require constructing edge subgraph posets of several graphs. But the number of graphs to be analysed is minimised by the following argument. Suppose G is a counter example to $\mathcal{P}_e$-reconstruction and $e(G) \leq 7$. If all its edge deleted subgraphs $G-e$ (by which we always mean $G-e$ minus the resulting isolated vertices) are $\mathcal{P}_e$-reconstructible, then G is also $\mathcal{P}_e$-reconstructible since graphs having 4 to 7 edges are edge reconstructible. Therefore, we assume that at least one of the

subgraphs $G-e$ is a counter example to $\mathcal{P}_e$-reconstruction. So we need to analyse only those graphs that are obtained by adding an edge in a smaller counter example.

Lemma 16. *Suppose all graphs on four or more edges are edge reconstructible. Suppose $\{G,H\}$ is a counter example to $\mathcal{P}_e$-reconstruction and $e(G) = e(H) \geq 7$ then $\{G,H\} = \{K_{1,n},\ (K_2)^n\}; n \geq 7$ or $\{G,H\} = \{(K_3)^p(K_{1,3})^q \prod_i F_i^{k_i},\ (K_{1,3})^r(K_3)^s \prod_i F_i^{k_i}\}$, where the components F_i are isomorphic to B_1 or (K_4/e) or K_4 or paths or cycles, and $(p,q) \neq (r,s)$ and $p+q = r+s$.*

Proof. This is proved by induction on $e(G)$. The base case $e(G) = 7$ follows from the 7-edge counter examples listed in Lemma 15. Let the claim be true when $7 \leq e(G) = e(H) < m$. Now let $\{G,H\}$ be a counter example to $\mathcal{P}_e$-reconstruction, and $e(G) = e(H) = m$. Therefore, there must be edge deleted subgraphs G_1 and H_1 of G and H, respectively, such that $\mathcal{P}_e(G_1) = \mathcal{P}_e(H_1)$ and $G_1 \ncong H_1$, (otherwise the edge reconstructibility would imply that $G \cong H$). By induction hypothesis we can assume that $G_1 \cong (K_3)^p(K_{1,3})^q \prod_i F_i^{k_i}$ and $H_1 \cong (K_3)^r(K_{1,3})^s \prod_i F_i^{k_i}$ where $(p,q) \neq (r,s)$ and $p+q = r+s$, and the components F_i are as described above.

All 7 edge counter examples contain the subgraph $K_{1,2}(K_2)^2$, and both G and H contain a 7-edge counter example as subgraph. Therefore, the vertices of $\mathcal{P}_e(G)$ that are labelled $K_{1,2}$, $(K_2)^2$, $K_{1,2}K_2$ and $(K_2)^3$ are recognised since all subgraphs of $K_{1,2}(K_2)^2$, are recognised. Therefore, all 3-edge subgraphs of G and H, (and of G_1 and H_1) except possibly $K_{1,3}$ and K_3, and all 4-edge subgraphs of G and H (and of G_1 and H_1), except possibly $K_{1,3}K_2$ and K_3K_2, are uniquely recognised. In particular, $\#(T_4 \xrightarrow{e} G) = \#(T_4 \xrightarrow{e} H)$, and all subgraphs of T_4 are unambiguously recognised. So if G contains T_4 as a subgraph then the vertex in the poset that corresponds to the subgraph $K_{1,3}$ is unambiguously recognised, and hence the vertex that corresponds to K_3 is recognised. Therefore, $\#(K_{1,3} \xrightarrow{e} G_1) = \#(K_{1,3} \xrightarrow{e} H_1)$ and $\#(K_3 \xrightarrow{e} G_1) = \#(K_3 \xrightarrow{e} H_1)$. But G_1 and H_1 contain different number of triangles and 3-stars. Therefore we assume that T_4 is not a subgraph of G or H.

Since T_4 is not a subgraph of G or H, the graphs B_1, (K_4/e) and K_4 can only appear as subgraphs of components on 4 vertices. Moreover, $\#(K_4 \xrightarrow{e} G) = \#(K_4 \xrightarrow{e} H)$, $\#((K_4/e) \xrightarrow{e} G) = \#((K_4/e) \xrightarrow{e} H)$, $\#(B_1 \xrightarrow{e} G) = \#(B_1 \xrightarrow{e} H)$, and $\#(K_3 \xrightarrow{e} G) + \#(K_{1,3} \xrightarrow{e} G) = \#(K_3 \xrightarrow{e} H) + \#(K_{1,3} \xrightarrow{e} H)$.

Suppose there is a component isomorphic to K_4. Suppose $G-e$ and $H-f$ are edge deleted subgraphs of G and H obtained by deleting an edge from a K_4, and $\mathcal{P}_e(G-e) \cong \mathcal{P}_e(H-f)$. This is recognised by counting the number of K_4's in G and $G-e$. In this case G is constructed by adding an edge to a component (K_4/e) in $G-e$ in a unique way. And H is similarly constructed by adding an edge to a component isomorphic to (K_4/e) in $H-f$ in a unique way. This implies that either $\{G-e, H-f\}$ is a counter example to $\mathcal{P}_e$-reconstruction of the type described by the lemma, and so is $\{G, H\}$ OR $G-e \cong H-f$ and $G \cong H$. Therefore, we assume that there is no component isomorphic to K_4.

Suppose there is a component isomorphic to (K_4/e). There is a unique edge in (K_4/e) that belongs to two 3-stars and two triangles. Suppose $G-e$ and $H-f$ are edge deleted subgraphs of G and H obtained by deleting such an edge, and $\mathcal{P}_e(G-e) \cong \mathcal{P}_e(H-f)$. This can be recognised from $\mathcal{P}_e(G)$ and $\mathcal{P}_e(G-e)$ by checking if $\#((K_4/e) \xrightarrow{e} G-e) = \#((K_4/e) \xrightarrow{e} G)-1$, and $\#(K_3 \xrightarrow{e} G-e)+\#(K_{1,3} \xrightarrow{e} G-e) = \#(K_3 \xrightarrow{e} G) + \#(K_{1,3} \xrightarrow{e} G) - 4$. Therefore, G is obtained by adding an edge in a component of $G-e$ that is isomorphic to C_4, and H is obtained similarly from $H-f$. Again, as in the case of K_4, either $\{G-e, H-f\}$ is a counter example to $\mathcal{P}_e$-reconstruction of the type described by the lemma, and so is $\{G, H\}$ OR $G-e \cong H-f$ and $G \cong H$. Therefore, we assume that there is no subgraph isomorphic to (K_4/e).

Suppose B_1 is a component of G. There are two edges in B_1 each of which belongs to exactly one P_4. Suppose $G-e$ and $H-f$ are edge deleted subgraphs of G and H obtained by deleting one such edge, and $\mathcal{P}_e(G-e) \cong \mathcal{P}_e(H-f)$. This is recognised from the conditions $\#(B_1 \xrightarrow{e} G-e) = \#(B_1 \xrightarrow{e} G) - 1$ and $\#(P_4 \xrightarrow{e} G-e) = \#(P_4 \xrightarrow{e} G)-1$. So G is obtained from $G-e$ by adding an edge in a component isomorphic to P_4 so as to create a component isomorphic to B_1. And H is constructed similarly from $H-f$. Therefore, either $\{G-e, H-f\}$ is a counter example to $\mathcal{P}_e$-reconstruction of the type described by the lemma, and so is $\{G, H\}$ OR $G-e \cong H-f$ and $G \cong H$. Therefore, we assume that B_1 is not a subgraph of G or H.

An edge e is an end edge of a component that is a path P_n if and only if $\#(P_k \xrightarrow{e} G) - \#(P_k \xrightarrow{e} G-e) = 1$ for $k \leq n$, and $\#(P_k \xrightarrow{e} G)-\#(P_k \xrightarrow{e} G-e) = 0$ for $k = n+1$. These conditions are recognised from $\mathcal{P}_e(G)$ and $\mathcal{P}_e(G-e)$. Again, G is obtained from $G-e$ by adding an edge at the end of a component isomorphic to P_{n-1} (or adding an isolated edge in case $n = 2$). And H is constructed similarly from

$H-f$. Therefore, either $\{G-e, H-f\}$ is a counter example to $\mathcal{P}_e$-reconstruction of the type described by the lemma, and so is $\{G,H\}$ OR $G-e \cong H-f$ and $G \cong H$. Therefore, we assume that paths are not components of G or H.

The graph G has a component isomorphic to a cycle $C_n; n \geq 4$ if and only if there is an edge e such that $\#(C_n \xrightarrow{e} G-e) = \#(C_n \xrightarrow{e} G) - 1$. This condition can be recognised from $\mathcal{P}_e(G)$ and $\mathcal{P}_e(G-e)$. Therefore, G is obtained from $G-e$ by joining the end vertices of a component isomorphic to P_n, and H is similarly constructed. Therefore, arguing as before, we assume that there are no components isomorphic to cycles. Therefore, all components of G and H are triangles or 3-stars, and their total number is identical in G and H. □

Theorem 5.

(1) *The edge reconstruction conjecture is true if and only if all graphs G except graphs T_5, C_4K_2, B_3 and B_4 can be constructed up to isomorphism from the pair $(\mathcal{P}_e(G), v(G))$.*

(2) *The edge reconstruction conjecture is true if and only if all graphs other than the following are reconstructible from their abstract edge subgraph posets:*
  - (a) $K_{1,n}$*;* $n \geq 2$ *and* $(K_2)^n$*;* $n \geq 2$,
  - (b) K_3*,* $K_{1,3}$ *and* $(K_2)^3$,
  - (c) P_4 *and* $K_{1,2}K_2$,
  - (d) P_4K_2 *and* T_4,
  - (e) C_4 *and* $(K_{1,2})^2$,
  - (f) C_4K_2 *and* T_5,
  - (g) P_6 *and* B_2,
  - (h) B_3 *and* B_4
  - (i) *graphs of the type* $(K_3)^p(K_{1,3})^q\prod_i F_i$ *where the components* F_i *are isomorphic to paths or cycles or* B_1 *or* (K_4/e) *or* K_4 *and* $p+q \neq 0$,

Proof. Suppose the edge reconstruction conjecture is true. It follows from Lemma 16 that, if $\{G,H\}$ is a counter example to $\mathcal{P}_e$-reconstruction and $e(G) = e(H) \geq 7$ then $v(G) \neq v(H)$. By Lemmas 13 and 14, $\{T_5, C_4K_2\}$ and $\{B_3, B_4\}$ are the only counter examples to $\mathcal{P}_e$-reconstruction for which additional knowledge of the number of vertices is not sufficient. This proves the *only if* part of (i).

If all graphs G on 7 or more edges can be constructed up to isomorphism from the pair $(\mathcal{P}_e(G), v(G))$ then edge reconstruction conjecture is true, since graphs on at most 6 edges (except the graphs $K_{1,2}$, $(K_2)^2$, $K_{1,3}$ and K_3) are known to be edge reconstructible. This proves the *if* part of (i).

The second part only reformulates the first part and summarises earlier lemmas. □

The class of graphs listed in the second part of Theorem 5 will now be denoted by $\underline{\mathcal{C}_p}$.

Theorem 6. *Trees (including forests for the purpose of this theorem) that are not in the class* $\underline{\mathcal{C}_p}$ *are reconstructible from their abstract edge subgraph posets.*

Proof. Since trees on four or more edges are edge reconstructible, an induction argument exactly similar to the one in Lemma 16 proves that trees that are not $\mathcal{P}_e$-reconstructible are of one of the following types: $K_{1,n}; n \geq 2$, $(K_2)^n; n \geq 2$, P_4, $K_{1,2}K_2$, $(K_{1,2})^2$, T_5, P_6 and $K_{1,3}^p \prod_i F_i$, where components F_i are paths. Such trees belong to $\underline{\mathcal{C}_p}$. □

Remark 6. We have not characterised the class $\mathcal{C}_p/\mathcal{C}$. We have only characterised the graphs that are not reconstructible from their abstract edge subgraph posets if the edge reconstruction conjecture is true. But if edge reconstruction conjecture is false then there may be other edge reconstructible graphs that are not $\mathcal{P}_e$-reconstructible. For example, if G and H are non-isomorphic connected graphs having the same edge deck then it is probably true that $G \uplus G$ and $H \uplus H$ are non-isomorphic, but have the same abstract edge subgraph poset.

Remark 7. We have seen earlier that if G and H are graphs such that $\mathcal{P}_e(G)$ appears as a down-set of a vertex in $\mathcal{P}_e(H)$ then it may be possible to unambiguously recognise from $\mathcal{P}_e(H)$ the vertex that corresponds to graph G even if G itself is not reconstructible from $\mathcal{P}_e(G)$. For example, $K_{1,3}$ and K_3 have isomorphic edge subgraph posets, but if they appear as subgraphs in another graph G which also has T_4 as a subgraph then the vertices in $\mathcal{P}_e(G)$ that correspond to $K_{1,3}$ and K_3 can be unambiguously recognised. Here the structure of $\mathcal{P}_e(T_4)$ resolves the ambiguity.

Now suppose that H is in $\mathcal{C}_p/\mathcal{C}$, and G is its counterpart such that $\mathcal{P}_e(G) \cong \mathcal{P}_e(H)$ and $G \ncong H$. Therefore, there is at least one edge deleted subgraph H_1 of H, and correspondingly a subgraph G_1 of G such that $\mathcal{P}_e(G_1) \cong \mathcal{P}_e(H_1)$ and $G_1 \ncong H_1$. But by argument similar to that for T_4, $K_{1,3}$ and K_3, it may be possible to recognise the vertices corresponding to G_1 and H_1 unambiguously. Therefore, there must be additional structural restrictions on $\mathcal{P}_e(G)$. Such a structural characterisation of graphs in $\mathcal{C}_p/\mathcal{C}$ and their edge subgraph posets might shed more light on edge reconstruction conjecture itself.

Remark 8. We can show that all graphs of the type $(K_3)^p(K_{1,3})^q\prod_i F_i$, where the components F_i are isomorphic to paths or cycles or B_1 or (K_4/e) or K_4 and $p+q\neq 0$, are in fact counter examples to $\mathcal{P}_e$-reconstruction. This will be done in the next version of this document.

Remark 9. The counter examples of the type $(K_3)^p(K_{1,3})^q\prod_i F_i$ are graphs which are not uniquely determined by their line graphs.

The following result is a weak form of Müller's result.

Theorem 7. *The edge reconstruction conjecture is true if and only if all graphs G such that $2^{e(G)-1}\leq v(G)!$ except the ones in the class $\underline{\mathcal{C}_p}$ are reconstructible from their abstract edge subgraph posets.*

Proof. If the edge reconstruction conjecture is true then by Theorem 5 all graphs not in $\underline{\mathcal{C}_p}$ are reconstructible from their abstract edge subgraph posets. They include graphs G not in $\underline{\mathcal{C}_p}$ that satisfy the bound $2^{e(G)-1}\leq v(G)!$.

If all graphs G not in $\underline{\mathcal{C}_p}$ that satisfy the above bound are reconstructible from their abstract edge subgraph posets then they are also edge reconstructible. The graphs that do not satisfy the above bound, that is graphs G for which $2^{e(G)-1}>v(G)!$, were proved to be edge reconstructible by Müller [8]. □

Remark 10. In other words, if all graphs G such that $2^{e(G)-1}\leq v(G)!$ except the ones in the class $\underline{\mathcal{C}_p}$ are reconstructible from their abstract edge subgraph posets then all other graphs (that is the ones for which $2^{e(G)-1}>v(G)!$ and not in class $\underline{\mathcal{C}_p}$) are reconstructible from their abstract edge subgraph posets.

6. Lovász's homomorphism cancellation laws

Let G and H be two graphs. Let $f: V(G)\to V(H)$ be such that if $\{x,y\}$ is an edge in G then $\{f(x),f(y)\}$ is an edge in H. Here x and y may be identical in case of loops. The map f is called a homomorphism from graph G to graph H. Let $hom(G\to H)$ denote the number of homomorphisms from G to H. Lovász [5] proved the following result.

Theorem 8. (Lovász 1971)

(1) *Let G_1 and G_2 be two simple graphs. Suppose for all graphs H, $hom(G_1\to H)=hom(G_2\to H)$. Then G_1 and G_2 are isomorphic.*
(2) *Let G_1 and G_2 be two simple graphs. Suppose for all graphs H, $hom(H\to G_1)=hom(H\to G_2)$. Then G_1 and G_2 are isomorphic.*

Interestingly, the above results are given in the chapter on Reconstruction in Lovász's problem book [7].

Here we seek a generalisation of the above results in the spirit of the study of reconstruction on the subgraph posets, and pose the following problem.

Conjecture 1. *Let $\mathcal{G}$ be the class of finite simple graphs. Let $\pi : \mathcal{G} \to \mathcal{G}$ be a bijection such that, for all graphs G and H in $\mathcal{G}$, $hom(G \to H) = hom(\pi(G) \to \pi(H))$. Then π is the identity map.*

In the following, I will show that Conjecture 1 is weaker than the edge reconstruction conjecture.

Lemma 17. *Let $\pi : \mathcal{G} \to \mathcal{G}$ be a bijection such that, for all graphs G and H in $\mathcal{G}$, $hom(G \to H) = hom(\pi(G) \to \pi(H))$. Then $v(G) = v(\pi(G))$ and $e(G) = e(\pi(G))$, and π is an identity map on all graphs on at most 4 vertices and also for all graphs having at most 3 edges.*

Proof. First we prove the result for the null graph Φ. Let $\pi(G) = \Phi$ and $\pi(\Phi) = H$ for some graphs G and H. Therefore, $hom(G \to \Phi) = hom(\pi(G) \to \pi(\Phi)) = hom(\Phi \to H)$. Since every map from Φ to H is a homomorphism, and there is only one such map, $hom(\Phi \to H) = 1$. Therefore, $hom(G \to \Phi) = 1$, which is possible only if $v(G) = 0$. Therefore, $\pi(\Phi) = \Phi$.

Next we prove that $\pi(K_1) = K_1$. Let $\pi(G) = K_1$ and $\pi(K_1) = H$ for some graphs G and H. Since, $\pi(\Phi) = \Phi$, G and H are non-null. Now, $hom(G \to K_1) = hom(\pi(G) \to \pi(K_1)) = hom(K_1 \to H) = v(H)$, while $hom(G \to K_1) = 1$ if $e(G) = 0$ and $hom(G \to K_1) = 0$ otherwise. Therefore, $v(H) = 1$, and, H being simple, $H \cong K_1$.

Now, $v(G) = hom(K_1 \to G) = hom(\pi(K_1) \to \pi(G)) = hom(K_1 \to \pi(G)) = v(\pi(G))$. So $v(G) = v(\pi(G))$. Since $hom(K_2 \to K_1) = 0$, $\pi(K_2)$ must be non-empty, therefore, $\pi(K_2) = K_2$, and $\pi((K_1)^2) = (K_1)^2$. Since $2e(G) = hom(K_2 \to G) = hom(K_2 \to \pi(G)) = 2e(\pi(G))$, we have $e(G) = e(\pi(G))$.

Now we are ready to prove the result for all graphs on at most 4 vertices. The result is true for all 3 vertex graphs and for 4 vertex graphs $(K_1)^4$, $K_2(K_1)^2$, $K_4 - e$ and K_4 by the preceding paragraph. Is it possible that $\pi((K_2)^2) = K_{1,2}K_1$ and $\pi(K_{1,2}K_1) = (K_2)^2$? It is ruled out by observing that $hom(K_{1,2} \to (K_2)^2) = 4$, while $hom(K_{1,2} \to K_{1,2}K_1) = 6$.

There are three mutually non-isomorphic graphs on 4 vertices and having 3 edges: they are P_4, K_3K_1 and $K_{1,3}$. Since $hom(K_3 \to K_3K_1) = 6$, $hom(K_3 \to P_4) = 0$, $hom(K_3 \to K_{1,3}) = 0$, we claim that

$\pi(K_3K_1) = K_3K_1$. Now we observe that $hom(K_{1,2} \to P_4) = 10$, while $hom(K_{1,2} \to K_{1,3}) = 12$. Therefore, $\pi(K_{1,3}) = K_{1,3}$ and $\pi(P_4) = P_4$.

There are two mutually non-isomorphic graphs on 4 vertices having 4 edges. They are B_1 (shown in Figure 6) and C_4. We claim that $\pi(B_1) = B_1$ and $\pi(C_4) = C_4$ by observing that $hom(K_3 \to B_1) = 6$, while $hom(K_3 \to C_4) = 0$.

Now to prove that the map π is identity map for graphs having at most 3 edges, we need to look at only the graphs $G_1 \cong K_{1,2}K_2(K_1)^n$ and $G_2 \cong (K_2)^3(K_1)^{n-1}$, which are both graphs on $n+5$ vertices. We claim that $\pi(G_1)$ cannot be G_2 by counting $\hom(P_3 \to G_1)$ and $\hom(P_3 \to G_2)$. □

Thus the known counter examples to edge reconstruction are not counter examples to Conjecture 1. Now we prove that Conjecture 1 is in fact weaker than the edge reconstruction conjecture.

Lemma 18. *The edge reconstruction conjecture implies Conjecture 1.*

Proof. Let π be a bijection from $\mathcal{G}$ to itself that satisfies $hom(G \to H) = hom(\pi(G) \to \pi(H))$. for all G and H in $\mathcal{G}$. We assume the edge reconstruction conjecture to be true, and prove Conjecture 1 by induction on the number of edges. The base case of induction is $e(G) = 3$ and is proved in Lemma 17. Suppose now that G is a graph such that $e(G) = m+1$ where $m \geq 3$, and suppose that for all graphs H having m or fewer edges, $\pi(H) = H$.

For graphs H and G, let $mon(H \to G)$ denote the number of monomorphisms (= one-to-one homomorphisms) from H to G. Let Θ be an equivalence relation on $V(H)$. Let H/Θ denote the graph obtained by identifying vertices in each equivalence class of Θ. Each homomorphism from H to G is a monomorphism from H/Θ to G for some equivalence relation Θ on $V(H)$. Therefore,

$$hom(H \to G) = \sum_{\Theta} mon(H/\Theta \to G), \tag{13}$$

where the summation is over all equivalence relations on $V(H)$. Following Lovász [7] (solution to Problem 20, Chapter 15), Equation (13) can be solved for $mon(H \to G)$ in terms of $hom(H/\Theta \to G)$.

Now let H be a graph on m or fewer edges. So $\pi(H) = H$. We solve Equation (13) for $mon(H \to \pi(G))$ in terms of $hom(H/\Theta \to \pi(G))$. Since H has m or fewer edges, each graph $H/_{\Theta}$ has m or fewer edges. Therefore, $hom(H/\Theta \to G) = hom(H/\Theta \to \pi(G))$ for all equivalence relations Θ on $V(H)$. Therefore, $mon(H \to G) = mon(H \to \pi(G))$. Therefore, G and $\pi(G)$ have the same edge decks. Now the edge reconstruction conjecture implies Conjecture 1. □

7. Discussion

We proved that the original reconstruction questions of Ulam and Harary are equivalent to reconstructibility of all graphs (except the listed counter examples) from their vertex and edge subgraph posets, respectively. But this does not mean that a graph G is (vertex or edge) reconstructible if and only if G can be constructed from its (vertex or edge) subgraph poset. For an individual graph, the reconstructibility from the poset is much more difficult. But if a class of graphs is closed under vertex or edge deletion (for example trees) then the reconstructibility from subgraphs also implies reconstructibility from the poset. We therefore take a point of view that a reconstruction result is "significant", provided it can be proved for vertex or edge subgraph posets. This of course does not mean that many well known reconstruction results for classes of graphs are worthless. They are of significance from the point of interesting graph theoretic techniques. But if a class of graphs is not closed under vertex or edge deletion then it perhaps means that a reconstructibility result for the class makes use of certain special properties of the class. For example, it was shown in [13] that if Ulam's conjecture were false then disconnected graphs would not be reconstructible from their induced subgraph posets, although disconnected graphs are known to satisfy Ulam's conjecture. In this sense the contributions of Tutte in reconstruction - his results on reconstructibility of various graph polynomials and the techniques he used - pass the test of significance. It is conceivable that Lovász's classic result on edge reconstruction [6] would pass the test of significance as well. But it seems unlikely that Nash-Williams' lemma or a variant would pass such a test. It is not even clear what form of Nash-William's lemma would explain the counter examples to $\mathcal{P}_e$-reconstruction. In this context I suggest the following problem.

Problem 1. Prove that if edge reconstruction conjecture is false then there are infinitely many graphs G such that $2^{e(G)-1} > v(G)!$ (preferably with $e(G) = \binom{v(G)}{2}/2$) that are not reconstructible from the edge subgraph poset.

Computing invariants and studying reconstruction conjectures on abstract posets have other potential applications. If an invariant can be computed from certain unlabelled poset associated with a graph then probably analogous invariant can be defined for arbitrary posets and lattices (with mild restrictions: for example, they have to have rank functions). This would allow generalising graph invariants in a

natural way to many other objects, provided posets or lattices can be associated with the objects in a natural way.

Acknowledgements. I am supported by the Allan Wilson Centre for Molecular Ecology and Evolution, New Zealand.

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